Voltage Ratio and Impedance Calculations for Coupled Inductors and Autotransformers

Jacques Audet Aug 2022
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Ref. Electronic Engineering by Alley & Atwood p. 324
Coupled Inductors
Ref. Wikipedia
https://en.wikipedia.org/wiki/Inductance#Mutual_inductance

General Equations for Coupled Inductors

$$\begin{pmatrix} V1 \\ V2 \end{pmatrix} = \begin{pmatrix} Z11 & Z12 \\ Z21 & Z22 \end{pmatrix} \cdot \begin{pmatrix} I1 \\ I2 \end{pmatrix}$$

Vx = voltages
Zxx = Impedances
Ix = currents

$$V1 = Z11 \cdot I1 + Z12 \cdot I2$$

$$V2 = Z21 \cdot I1 + Z22 \cdot I2$$

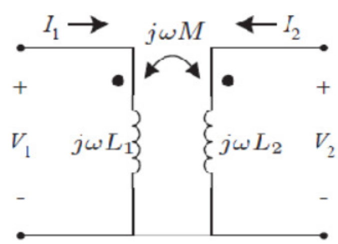
Calculation of Zin by setting V2 = 0

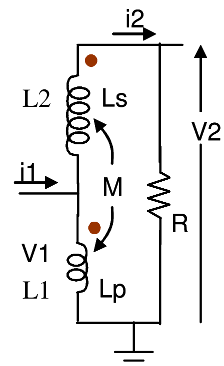
$$0 = Z21 \cdot I1 + Z22 \cdot I2$$

$$I2 = -\frac{I1 \cdot Z21}{Z22}$$

$$V1 = Z11 \cdot I1 + Z12 \cdot I2$$

$$V1 = \frac{I1 \cdot (Z11 \cdot Z22 - Z12 \cdot Z21)}{Z22}$$





$$Zin1 = \frac{Z11 \cdot Z22 - Z12 \cdot Z21}{Z22}$$

General equation for Zin at port 1

For separate coupled inductors:

$$Z11 = j \cdot w \cdot L1$$

$$Z12 = Z21 = j \cdot w \cdot M$$

$$Z22 = R + j \cdot w \cdot L2$$

Where M is the mutual inductance for both cases

For both cases, with K = coupling coeff.

$$M = K \cdot \sqrt{L1 \cdot L2}$$

For the Autotransformer:

$$Z11 = j \cdot w \cdot L1$$

$$Z12 = Z21 = -j \cdot w \cdot (L1 + M)$$

$$Z22 = R + j \cdot w \cdot L$$

R = Load resistance at secondary

Notes:
The negative sign of Z12 and Z21 is used since i2 flows out of the polarity mark

$$L = L1 + L2 + 2 \cdot M$$

Inductance at output

Exemple of Calculation of Zin1: R is on the secondary side (V2) for the autotransformer

$L1 := 3 \qquad L2 := 36 \cdot L1 \qquad R := 2450 \qquad K := 0.95 \qquad M := K \cdot \sqrt{L1 \cdot L2} \qquad L := L1 + L2 + 2 \cdot M$

Turns ratio = 6 so L2 / L1 = 36

$f := 1, 1.1 .. 100$ F in MHz and L, M in uH

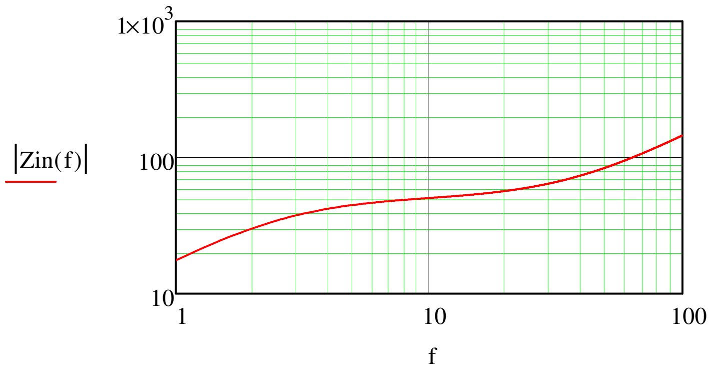
$w(f) := 2 \cdot \pi \cdot f$

$Z11(f) := j \cdot w(f) \cdot L1$
$Z12(f) := -j \cdot w(f) \cdot (L1 + M)$
$Z21(f) := Z12(f)$
$Z22(f) := R + j \cdot w(f) \cdot L$

NOTE :
Interchanging L1 and L in Z11 and Z22 will calculate Zin2 (secondary side)
with R on the primary side (V1)

$$Zin(f) := \frac{Z11(f) \cdot Z22(f) - Z12(f) \cdot Z21(f)}{Z22(f)}$$

General equation for Zin at port 1 (input)



$|Zin(1)| = 17.671$

$|Zin(10)| = 50.639$

$|Zin(100)| = 145.717$

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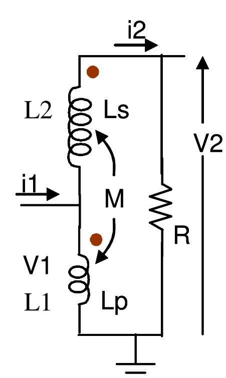
Calculation of Av, the voltage transfer ratio

$V1 = Z11 \cdot I1 + Z12 \cdot I2$

$V2 = Z21 \cdot I1 + Z22 \cdot I2$

With V2=0 Calculate Av = V2 / V1
R is on the sec. (V2)



$0 = Z21 \cdot I1 + Z22 \cdot I2$

$I1 = -\frac{I2 \cdot Z22}{Z21}$ Solve for I1

$V1 = Z11 \cdot I1 + Z12 \cdot I2$

$V1 = -\frac{I2 \cdot (Z11 \cdot Z22 - Z12 \cdot Z21)}{Z21}$

Recall that: $Av = \frac{I2}{V1} \cdot R$

$$Av = \frac{Z21 \cdot R}{Z12 \cdot Z21 - Z11 \cdot Z22}$$

General equation for Av = V2 / V1. This is NOT insertion loss !

Example

$L1 := 3$ $\quad L2 := 36 \cdot L1$ $\quad R := 2450$ $\quad K := 0.95$ $\quad M := K \cdot \sqrt{L1 \cdot L2}$ $\quad L := L1 + L2 + 2 \cdot M$

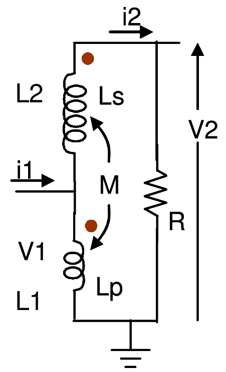
$Z11(f) := j \cdot w(f) \cdot L1$

$Z12(f) := -j \cdot w(f) \cdot (L1 + M)$

$Z21(f) := Z12(f)$

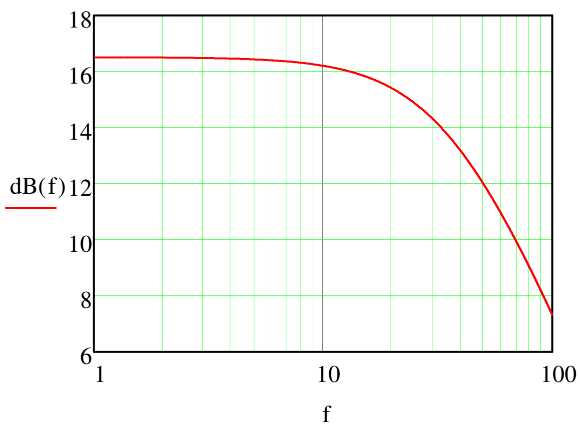
$Z22(f) := R + j \cdot w(f) \cdot L$

NOTE :
Interchanging L1 and L in Z11 and Z22 will calculate V1 / V2 with R on the primary side (V1)



$$Av(f) := \frac{Z21(f) \cdot R}{Z12(f) \cdot Z21(f) - Z11(f) \cdot Z22(f)}$$

$$dB(f) := 20 \cdot \log(|Av(f)|)$$



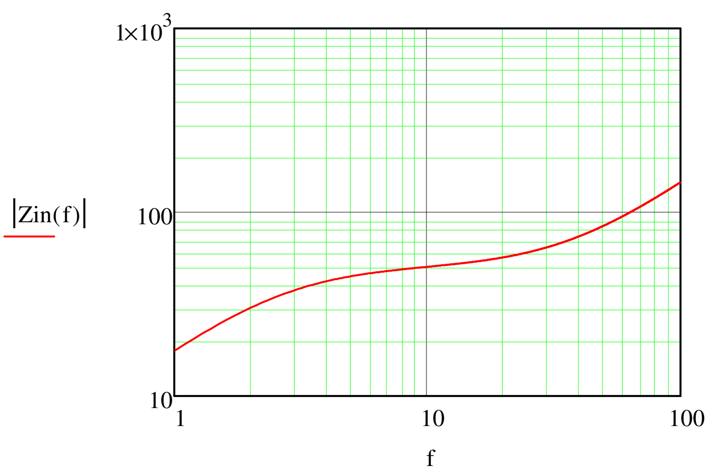
$|Av(1)| = 6.698$

$|Av(10)| = 6.468$

$|Av(100)| = 2.327$

Zin and SWR Calculation at Input

$$Zin(f) := \frac{Z11(f) \cdot Z22(f) - Z12(f) \cdot Z21(f)}{Z22(f)}$$



$|Zin(1)| = 17.671$

$|Zin(10)| = 50.639$

$|Zin(100)| = 145.717$

$Zin(1) = 5.717 + 16.721i$

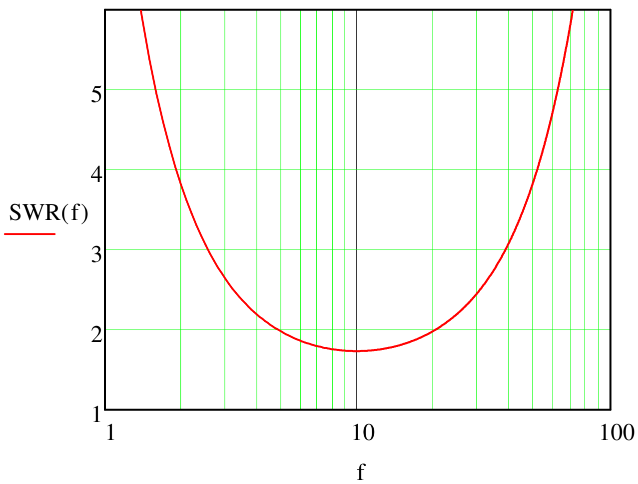
$Zin(10) = 43.791 + 25.43i$

$Zin(100) = 46.915 + 137.958i$

SWR Calculation based on Zin, at input

$Zo := 50$

$$SWR(f) = \frac{1 + \left|\frac{Zin(f) - Zo}{Zin(f) + Zo}\right|}{1 - \left|\frac{Zin(f) - Zo}{Zin(f) + Zo}\right|} \qquad SWR(f) := \frac{2 \cdot |Zo + Zin(f)|}{|Zo + Zin(f)| - |Zo - Zin(f)|} - 1$$



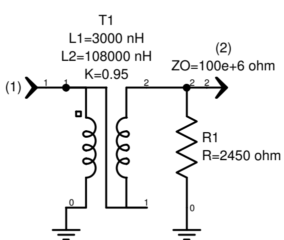
$SWR(3.5) = 2.372$

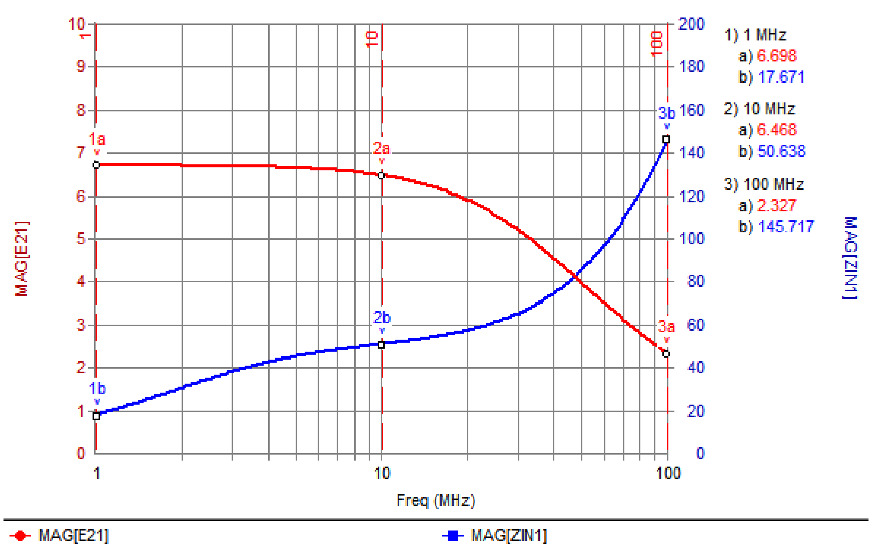
$SWR(10) = 1.737$

$SWR(29) = 2.422$

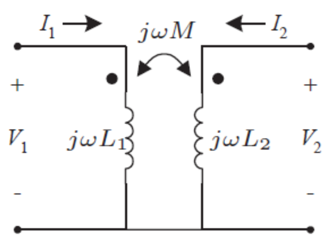
Ref: IndependantCoupledCoils-VS-Tapped coil-1.wsp

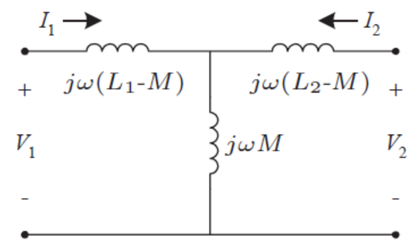
Simulations





Phasor Analysis: T-Equivalent





Frequency Domain (Phasors)

$$\begin{aligned} V_1 &= j\omega L_1 I_1 + j\omega M I_2 \\ V_2 &= j\omega M I_2 + j\omega L_2 I_2 \end{aligned}$$

- The T-equivalent is only valid if bottom terminals are connected

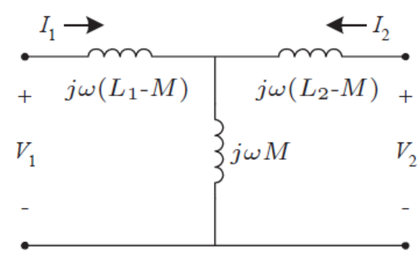
Portland State University   ECE 221   Magnetically Coupled Circuits   Ver. 1.32

Equivalent Tee circuit for Independant Inductors

$Z11 = j \cdot w \cdot L1$

$Z12 = Z21 = j \cdot w \cdot M$

$Z22 = R + j \cdot w \cdot L2$



Equivalent Tee Circuit for the Autotransformer

For the Autotransformer:

$Z11 = j \cdot w \cdot L1$

$Z12 = Z21 = -j \cdot w \cdot (L1 + M)$

$Z22 = R + j \cdot w \cdot L$

From comparisons with the independant inductor Tee circuit above

La = L1 - (L1 + M)

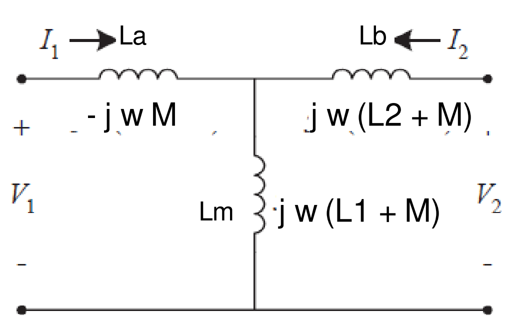
Lb = L - (L1+M) = L2 + M

Lm = L1 + M

M = K * sqrt(L1 * L2)

L = L1 + L2 + 2 * M

Autotransformer equivalent Tee circuit



Example, using the same values as in page 3:

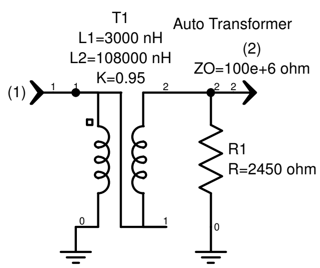
$L1 := 3 \qquad L2 := 36 \cdot L1 \qquad R := 2450 \qquad K := 0.95 \qquad M := K \cdot \sqrt{L1 \cdot L2} \qquad L := L1 + L2 + 2 \cdot M$

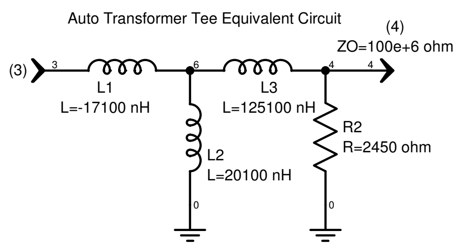
M and L can be calculated:

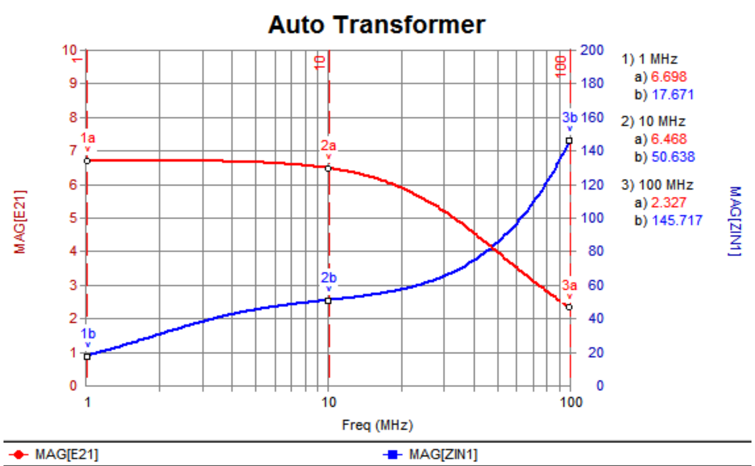
$M := K \cdot \sqrt{L1 \cdot L2} = 17.1 \qquad L := L1 + L2 + 2 \cdot M = 145.2$

Tee Equivalent circuit:

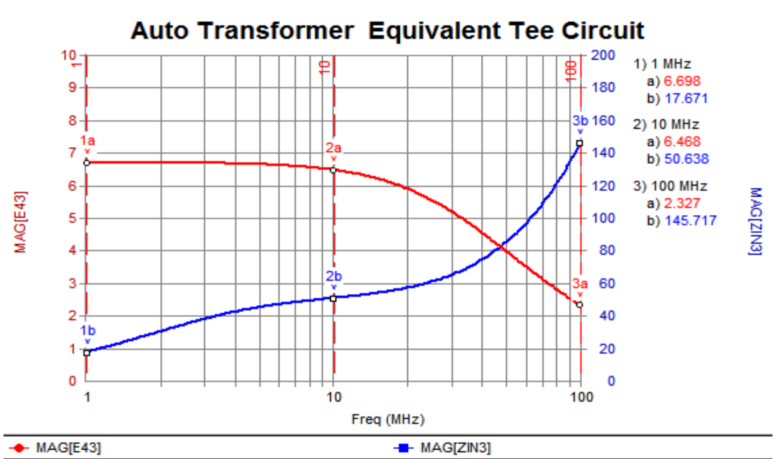
$La := -M = -17.1 \qquad Lb := L2 + M = 125.1 \qquad Lm := L1 + M = 20.1$







AutoTransformer Simulation



AutoTransformer Tee Equivalent Circuit Simulation

Voltage Gain Variation vs Coupling Coefficient K

With K = 1 (perfect coupling) the low frequency gain = 7.00
With K = 0.95 the low frequency gain = 6.698 (dotted curves)
With K = 0.90 the low frequency gain = 6.391 (solid curves)